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Impulsive periodic boundary value problems of first-order differential equations ☆

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Abstract

In this paper, by using Schaeffer's theorem, we prove new existence theorems for a nonlinear periodic boundary value problem of first-order differential equations with impulses. Our results improve and generalize some known results.

Keywords: Impulsive differential equation; Periodic boundary value problem; Green's function; Schaeffer's theorem

1. Introduction

The theory of impulsive differential equations has been emerging as an important area of investigations in recent years (see, [1–3,12]). Differential equations involving impulsive effects occurs in many applications: population dynamics, biological systems, industrial

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robotic, optimal control, etc. It should be noted that many publications dealt with periodic boundary value problem (PBVP for short) of impulsive differential equations (see, [4,5,7–10,12] etc). In [4,9,10], the authors studied the PBVP of nonlinear problem and got some new results. In [7], the investigator obtained some results by using the method of upper and lower solutions coupled with the monotone iterative technique and comparison principle.

In this paper, we deal with a periodic boundary value problem for differential equations with impulsive effects of the form

$$u'(t) + \lambda u(t) = f\big(t, u(t)\big), \quad t \in J,\ t \neq t_k,\ k = 1, \ldots, p, \tag{1.1}$$

$$u\big(t_k^+\big) = u\big(t_k^-\big) + I_k\big(u(t_k)\big), \quad k = 1, \ldots, p, \tag{1.2}$$

$$u(0) = u(T), \tag{1.3}$$

where $\lambda \in R$ and $\lambda \neq 0$, $J = [0, T]$, $T > 0$, $0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = T$, $I_k \in C(R, R)$, $k = 1, \ldots, p$, and $f : J \times R \to R$ is continuous at every point $(t, u) \in J_0 \times R$, $J_0 = J - \{t_1, \ldots, t_p\}$, $f(t_k^+, u)$ and $f(t_k^-, u)$ exist, $f(t_k^-, u) = f(t_k, u)$.

Let $PC(J) = \{u$: u is a map from J into R such that $u(t)$ is continuous at $t \neq t_k$, and left continuous at $t = t_k$, and the right limit $u(t_k^+)$ exists for $k = 1, 2, \ldots, p\}$. Evidently, $PC(J)$ is a Banach space with the norm

$$\|u\|_{PC} = \sup_{t \in J}\big\{\big|u(t)\big|\big\}.$$

Note that $PC(J)$ is equivalent to $\prod_{k=0}^{p} C[t_k, t_{k+1}]$. Analogously define the Banach space

$$PC^1(J) = \Big\{u \in PC(J)\colon\ u|_{(t_k, t_{k+1})} \in C^1(t_k, t_{k+1}),\\ \text{there exist } u'\big(t_k^-\big) \text{ and } u'\big(t_k^+\big),\ k = 1, 2, \ldots, p\Big\}$$

with the norm

$$\|u\|_{PC^1} = \max\big\{\|u\|_{PC}, \|u'\|_{PC}\big\}.$$

In this paper, we shall complement and improve some results in [4], and obtain some new results. This paper is organized as follows. Firstly, in Section 2, we prove the existence of solution to the "linear problem" of impulsive differential equations, which is a base of nonlinear problem. Then we also obtain, in Section 3, some new results on existence of solutions of PBVP for the nonlinear problem. Finally we work through an example to illustrate our results.

We will need the following lemma.

Lemma 1.1 *(Schaeffer's theorem [11]). Assume S be a normed linear space, and let operator $F : S \to S$ be compact. Define*

$$H(F) = \big\{x \in S\colon\ x = \mu F(x),\ \mu \in (0, 1)\big\}.$$

Then either

(i) *set $H(F)$ is unbounded*; *or*
(ii) *operator F has a fixed point in S.*

2. Linear problem

In this section, we consider the "linear problem"

$$u'(t) + \lambda u(t) = \sigma(t), \quad t \in J_0, \tag{2.1}$$

$$u(t_k^+) = u(t_k^-) + I_k(u(t_k)), \quad k = 1, \dots, p, \tag{2.2}$$

$$u(0) = u(T), \tag{2.3}$$

where $\sigma \in PC(J)$ and $I_k \in C(R, R)$, $k = 1, \dots, p$.

For short, we shall refer to (2.1)–(2.3) as (LP). Note that (LP) is not really a linear problem since the impulsive functions I_k are not necessarily linear. However, if I_k, $k = 1, \dots, p$, are linear, then (LP) is a linear impulsive problem.

Firstly we present the following lemma (see Lemma 2.1 in [4]).

Lemma 2.1. *$u \in PC^1(J)$ is a solution of* (LP) *if and only if u is a solution of the integral equation*

$$u(t) = \int_0^T g(t,s)\sigma(s)\,ds + \sum_{k=1}^{p} g(t,t_k) I_k(u(t_k)), \quad t \in J, \tag{2.4}$$

where

$$g(t,s) = \frac{1}{1 - e^{-\lambda T}} \begin{cases} e^{-\lambda(t-s)}, & 0 \leqslant s \leqslant t \leqslant T, \\ e^{-\lambda(T+t-s)}, & 0 \leqslant t < s \leqslant T. \end{cases} \tag{2.5}$$

We now define the operator $A : PC(J) \to PC(J)$ by

$$Au = \int_0^T g(t,s)\sigma(s)\,ds + \sum_{k=1}^{p} g(t,t_k) I_k(u(t_k)).$$

Hence Eq. (2.4) can be viewed as the operator equation

$$u = Au. \tag{2.6}$$

An abstract criterion for the solvability of (LP) is given in the following result which is an immediate consequence of Lemma 2.1.

Lemma 2.2. *u is a solution of* (LP) *if and only if u is a fixed point of Eq.* (2.6).

Now we discuss the existence of solutions for the problem (LP).

Theorem 2.1. *Suppose that there exist constants l_k, $k = 1, \dots, p$, such that*

$$|I_k(x) - I_k(y)| \leqslant l_k |x - y|, \quad x, y \in R. \tag{2.7}$$

If

$$e^{|\lambda| T} \sum_{k=1}^{p} l_k < |1 - e^{-\lambda T}|, \tag{2.8}$$

then the problem (LP) *has a unique solution for any $\sigma \in PC(J)$.*

Proof. Let $u, v \in PC(J)$ and $t \in J$. We have that

$$\begin{aligned}\big|(Au)(t)-(Av)(t)\big| &= \left|\sum_{k=1}^{p} g(t,t_k)\big[I_k\big(u(t_k)\big)-I_k\big(v(t_k)\big)\big]\right| \\ &\leqslant \sum_{k=1}^{p}\sup\big\{\big|g(t,t_k)\big|:\ t\in J\big\}l_k\big|u(t_k)-v(t_k)\big| \\ &\leqslant \frac{e^{|\lambda|T}}{|1-e^{-\lambda T}|}\|u-v\|\sum_{k=1}^{p}l_k,\end{aligned}$$

i.e.,

$$\|Au-Av\|\leqslant \frac{e^{|\lambda|T}\sum_{k=1}^{p}l_k}{|1-e^{-\lambda T}|}\|u-v\|,$$

using (2.8) and Banach's fixed point theorem, A has a unique fixed point which is the unique solution of (LP). This theorem is proved. □

Remark 2.1. A similar result was given in [4] (see Theorem 2.1 in [4]), where the condition

$$\sum_{j=1}^{p} l_j < \big|1-e^{-\lambda T}\big|, \tag{2.9}$$

was used. However, Theorem 2.1 in [4] only is valid as $\lambda > 0$ but not as $\lambda < 0$. The following is an illustrative example.

Example 1. Consider problem (LP) with $\lambda = -1$, $k = 1$ and $I_1(x) = cx$, $c = e^{-T} - 1$, and $\sigma \equiv 1$. We get

$$u(t)=\begin{cases} u(0)e^{t}+e^{t}-1, & t\in[0,t_1],\\ u(t_1^+)e^{t-t_1}+e^{t-t_1}-1, & t\in(t_1,T].\end{cases}$$

Thus, u satisfies the periodic condition (2.3) if and only if

$$u(0)=u(T)=(c+1)u(0)e^{T}+1-e^{-t_1}+e^{T-t_1}-1,$$

i.e.,

$$u(0)=u(0)+e^{-t_1}\big(e^{T}-1\big).$$

This condition is not satisfied for any initial condition $u(0)$ since $e^{-t_1}(e^T - 1) > 0$. Thus the problem (LP) has no solution. We note that condition (2.8) is not satisfied since

$$\frac{e^{|\lambda|T}\sum_{k=1}^{p}l_k}{|1-e^{-\lambda T}|}=\frac{e^{T}(1-e^{-T})}{e^{T}-1}=\frac{e^{T}-1}{e^{T}-1}=1,$$

where we set $l_1 = |e^{-T} - 1|$. However, since

$$\frac{\sum_{k=1}^{p}l_k}{|1-e^{-\lambda T}|}=\frac{|e^{-T}-1|}{|1-e^{T}|}<1,$$

we see that condition (2.9) is satisfied, showing that Theorem 2.1 in [4] is not valid for $\lambda < 0$.

3. Nonlinear problem

To study the nonlinear impulsive problem (1.1)–(1.3), define the operator $B : PC(J) \to PC(J)$ by

$$Bu = \int_0^T g(t,s) f\big(s, u(s)\big)\, ds + \sum_{k=1}^p g(t, t_k) I_k\big(u(t_k)\big).$$

We refer to (1.1)–(1.3) as (NP). Analogously, the problem (NP) has solutions if and only if the following operator equation has fixed points

$$u = Bu.$$

Lemma 3.1. *B is compact.*

Proof. (i) Let $D \subset PC(J)$ be a bounded set. It follows that

$$\big|f(t,u)\big| \leqslant M, \quad \big|I_k(u)\big| \leqslant M_k, \quad \forall u \in D,\ t \in J,$$

where M, M_k are constants. Hence

$$\begin{aligned}\big|(Bu)(t)\big| &\leqslant M \int_0^T \big|g(t,s)\big|\, ds + \sum_{k=1}^p \big|g(t,t_k)\big|\big|I_k(u)\big| \\ &\leqslant \frac{M}{|\lambda|} + \frac{e^{|\lambda| T}}{|1 - e^{-\lambda T}|} \sum_{k=1}^p M_k.\end{aligned}$$

We obtain

$$\|Bu\| \leqslant \frac{M}{|\lambda|} + \frac{e^{|\lambda| T}}{|1 - e^{-\lambda T}|} \sum_{k=1}^p M_k.$$

This implies that $B(D)$ is uniformly bounded.

(ii) For any $\bar{t}, \underline{t} \in (t_{k-1}, t_k] \cap J$, $u \in D$, we have

$$\begin{aligned}&\big|(Bu)(\bar{t}) - (Bu)(\underline{t})\big| \\ &\quad\leqslant \left| \int_0^{\bar{t}} f(s,u) e^{-\lambda(\bar{t}-s)}\, ds - \int_0^{\underline{t}} f(s,u) e^{-\lambda(\underline{t}-s)}\, ds \right| \\ &\qquad + \left| \sum_{0<t_k<\bar{t}} I_k(u) e^{-\lambda(\bar{t}-t_k)} - \sum_{0<t_k<\underline{t}} I_k(u) e^{-\lambda(\underline{t}-t_k)} \right| \\ &\qquad + \frac{e^{-\lambda T}}{|1 - e^{-\lambda T}|} \int_0^T \big| f(s,u)\big(e^{-\lambda(\bar{t}-s)} - e^{-\lambda(\underline{t}-s)}\big)\big|\, ds\end{aligned}$$

$$+ \frac{e^{-\lambda T}}{|1-e^{-\lambda T}|}\left|\sum_{k=1}^{p} I_k(u)\left(e^{-\lambda(\bar{t}-t_k)} - e^{-\lambda(\underline{t}-t_k)}\right)\right|$$
$$\leqslant \left|\int_0^{\bar{t}} f(s,u)e^{-\lambda(\bar{t}-s)}\,ds - \int_0^{\underline{t}} f(s,u)e^{-\lambda(\underline{t}-s)}\,ds\right|$$
$$+ \sum_{0<t_k<\bar{t}} M_k\left|e^{-\lambda(\bar{t}-t_k)} - e^{-\lambda(\underline{t}-t_k)}\right|$$
$$+ \frac{e^{-\lambda T}}{|1-e^{-\lambda T}|}\left[M\int_0^T \left|e^{-\lambda(\bar{t}-s)} - e^{-\lambda(\underline{t}-s)}\right|ds + \sum_{k=1}^{p} M_k\left|e^{-\lambda(\bar{t}-t_k)} - e^{-\lambda(\underline{t}-t_k)}\right|\right],$$

which implies

$$\left|(Bu)(\bar{t}) - (Bu)(\underline{t})\right| \to 0, \quad \text{as } |\bar{t}-\underline{t}| \to 0.$$

Hence $B(D)$ is quasiequicontinuous and operator B is compact by Lemma 2.4 in [12]. This proof is complete. □

Motivated by [4], we can obtain new existence results.

Theorem 3.1. *Assume that there exist constants l and c_k such that*

$$\left|f(t,u)\right| \leqslant l|u|, \quad l \geqslant 0,$$
$$\left|I_k(u)\right| \leqslant c_k, \quad k = 1,\ldots,p,$$

and

$$\frac{l}{|\lambda|} < 1,$$

hold. Then the problem (NP) *has at least one solution.*

Proof. Let $u \in PC(J)$, $t \in J$. We consider the operator equation

$$u = \mu Bu, \quad \mu \in (0,1). \tag{3.1}$$

If u is a solution of Eq. (3.1), for $t \in J$, we have that

$$\left|u(t)\right| \leqslant \mu \int_0^T \left|g(t,s)f\left(s,u(s)\right)\right|ds + \mu\sum_{k=1}^{p}\left|g(t,t_k)I_k\left(u(t_k)\right)\right|$$
$$\leqslant \frac{\mu l\|u\|}{|\lambda|} + \frac{\mu e^{|\lambda|T}\sum_{k=1}^{p} c_k}{|1-e^{-\lambda T}|},$$

hence

$$\|u\| \leqslant \frac{\mu l\|u\|}{|\lambda|} + \frac{\mu e^{|\lambda|T}\sum_{k=1}^{p} c_k}{|1-e^{-\lambda T}|}$$
$$\leqslant \frac{l\|u\|}{|\lambda|} + \frac{e^{|\lambda|T}\sum_{k=1}^{p} c_k}{|1-e^{-\lambda T}|},$$

this and $\frac{l}{|\lambda|} < 1$ imply

$$\|u\| \leqslant \frac{e^{|\lambda| T} \sum_{k=1}^{p} c_k}{(1 - \frac{l}{|\lambda|})|1 - e^{-\lambda T}|}.$$

This shows that all the solutions of (3.1) are bounded independently of $\mu \in (0, 1)$. Using Lemmas 1.1 and 3.1, we obtain that B has a fixed point. This proof is complete. □

Analogously we can prove the following result.

Theorem 3.2. *Assume that*

(i) $|f(t, u)| \leqslant c$, $|I_k(u)| \leqslant l_k|u|$, $t \in J$, $u \in R$, $k = 1, \ldots, p$, *where* c *and* l_k *are constants,*
(ii) $e^{|\lambda| T} \sum_{k=1}^{p} l_k < |1 - e^{-\lambda T}|$.

Then the problem (NP) *is solvable.*

However, if $I_k(x)$ are linear, i.e., $I_k(x) = l_k x$, $k = 1, 2, \ldots, p$, l_k constants, then the condition (ii) of Theorem 3.2 is not needed and we have the following improvement.

Theorem 3.3. *Assume* $I_k(x) = l_k x$, $k = 1, 2, \ldots, p$, l_k *are constants. If the following conditions hold*

(i) $|f(t, u)| \leqslant c$ *for* $(t, u) \in J \times R$, *where* c *is a constant,*
(ii) $\prod_{k=1}^{p} b_k \neq e^{\lambda T}$, *where* $b_k = l_k + 1$,

then the problem (NP) *has at least one solution.*

Proof. In this case, (NP) becomes

$$\begin{aligned} & u'(t) + \lambda u(t) = f(t, u), \quad t \in J_0, \\ & u(t_k^+) = b_k u(t_k), \quad k = 1, 2, \ldots, p, \\ & u(0) = u(T). \end{aligned} \tag{3.2}$$

If $b_k = 0$, then $u(t_k^+) = 0$ for some k. In a similar way of the proof of Lemma 3.1 in [6], we can show (3.2) has at least one solution.

Next, we assume $b_k \neq 0$ for $k = 1, 2, \ldots, p$. Let $u(t)$ be any solution of (3.2). Set $y(t) = u(t) \prod_{0 \leqslant t_k < t} b_k^{-1}$ for $t > 0$, then for all $k \geqslant 1$, we have

$$\begin{aligned} y(t_k^+) &= u(t_k^+) \prod_{0 \leqslant t_i \leqslant t_k} b_i^{-1} = u(t_k) b_k \prod_{0 \leqslant t_i \leqslant t_k} b_i^{-1} \\ &= u(t_k) \prod_{0 \leqslant t_i < t_k} b_i^{-1} = y(t_k). \end{aligned}$$

Thus $y(t)$ is continuous on J. Furthermore, we see that $y(t)$ satisfies

$$y'(t)+\lambda y(t)=f\Bigl(t,y(t)\prod_{0\leqslant t_k<t}b_k\Bigr)\prod_{0\leqslant t_k<t}b_k^{-1},$$
$$y(0)=y(T)\prod_{k=1}^{p}b_k. \tag{3.3}$$

Analogously, if $y(t)$ is a solution of (3.3), then $u(t)=y(t)\prod_{0\leqslant t_k<t}b_k$ satisfies (3.2). Set

$$F\bigl(t,y(t)\bigr)=f\Bigl(t,y(t)\prod_{0\leqslant t_k<t}b_k\Bigr)\prod_{0\leqslant t_k<t}b_k^{-1}.$$

It follows that (3.3) has a solution if and only if the integral equation

$$y(t)=\int_0^T G(t,s)F\bigl(s,y(s)\bigr)\,ds$$

is solvable. Here,

$$G(t,s)=\frac{1}{e^{\lambda T}-\prod_{k=1}^{p}b_k}\begin{cases}e^{\lambda(T-t+s)}, & 0\leqslant s\leqslant t\leqslant T,\\ \prod_{k=1}^{p}b_k e^{\lambda(s-t)}, & 0\leqslant t<s\leqslant T.\end{cases}$$

Define the operator $B^*:C(J)\to C(J)$ by

$$B^*y=\int_0^T G(t,s)F\bigl(s,y(s)\bigr)\,ds.$$

Hence (3.3) is equivalent to $y=B^*y$. It is easy to show that B^* is compact. We consider the equation

$$y=\mu B^*y,\quad \mu\in(0,1). \tag{3.4}$$

If y is a solution of Eq. (3.4) for $t\in J$, then

$$\bigl|y(t)\bigr|\leqslant\int_0^T\bigl|G(t,s)\bigr|\bigl|F\bigl(s,y(s)\bigr)\bigr|\,ds$$
$$\leqslant c_1\int_0^T\bigl|G(t,s)\bigr|\,ds\leqslant c_1c_2,$$

where $c_1=\sup\{c|\prod_{0\leqslant t_k<t}b_k^{-1}|,\ t\in J\}$, $c_2=\sup\{\int_0^T|G(t,s)|\,ds,\ t\in J\}$. This implies that

$$\|y\|\leqslant c_1c_2,$$

and shows that all the solutions of (3.4) are bounded independently of $\mu\in(0,1)$, so B^* has a fixed point. Furthermore, Eq. (3.2) has at least one solution. □

Theorem 3.4. *Assume the following conditions hold:*

(i) $\lim_{|u|\to\infty} \frac{f(t,u)}{u} = 0$,
(ii) $\lim_{|u|\to\infty} \frac{I_k(u)}{u} = 0$, $k = 1, \ldots, p$.

Then the problem (NP) *is solvable.*

Proof. Let $u \in PC(J)$ and $t \in J$, we consider

$$u = \mu Bu, \quad \mu \in (0,1). \tag{3.5}$$

If the solutions of (3.5) are not bounded, then there exist sequences

$$\begin{aligned} &\{u_n\}_{n=1}^{\infty}, \quad u_n \in PC(J),\ \|u_n\| \geqslant n, \\ &\{\mu_n\}_{n=1}^{\infty}, \quad \mu_n \in (0,1), \end{aligned}$$

with

$$\begin{aligned} &u_n'(t) + \lambda u_n(t) = \mu_n f\big(t, u_n(t)\big), \\ &u_n\big(t_k^+\big) = u_n\big(t_k^-\big) + \mu_n I_k\big(u_n(t_k)\big), \quad k = 1, \ldots, p, \\ &u_n(0) = u_n(T). \end{aligned} \tag{3.6}$$

Now we let $v_n = \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$. By (3.6), we know that v_n satisfies

$$\begin{aligned} &v_n'(t) + \lambda v_n(t) = \mu_n \frac{f(t, u_n(t))}{\|u_n\|}, \\ &v_n\big(t_k^+\big) = v_n\big(t_k^-\big) + \mu_n \frac{I_k(u_n(t_k))}{\|u_n\|}, \quad k = 1, \ldots, p, \\ &v_n(0) = v_n(T). \end{aligned}$$

Set $\sigma_n(t) = \mu_n \frac{f(t,u_n(t))}{\|u_n\|}$, $\theta_{n,k} = \mu_n \frac{I_k(u_n(t_k))}{\|u_n\|}$. In view of Lemma 2.1, we get

$$v_n(t) = \int_0^T g(t,s)\sigma_n(s)\,ds + \sum_{k=1}^{p} g(t,t_k)\theta_{n,k}, \quad t \in J.$$

From conditions (i) and (ii), we have

$$\big|\sigma_n(t)\big| \leqslant \left|\frac{f(t,u_n)}{\|u_n\|}\right| \to 0 \quad \text{as } n \to \infty,$$

and

$$|\theta_{n,k}| \leqslant \left|\frac{I_k(u_n(t_k))}{\|u_n\|}\right| \to 0 \quad \text{as } n \to \infty,\ k = 1, \ldots, p.$$

Hence

$$\big|v_n(t)\big| \leqslant \int_0^T \big|g(t,s)\big|\big|\sigma_n(s)\big|\,ds + \sum_{k=1}^{p} \big|g(t,t_k)\big||\theta_{n,k}| \to 0, \quad \forall t \in J,\ \text{as } n \to \infty.$$

From this we deduce that $\{v_n\}_{n=1}^{\infty} \to 0$, which is a contradiction to the fact that $\|v_n\| = 1$. In view of Schaeffer's theorem, the problem (NP) has at least one solution. This proof is complete. □

Finally, by using Theorem 3.4, we can immediately get the following consequence.

Corollary 1 *(Bounded case, Theorem 3.1 in [4]). Assume that the nonlinearity f is bounded and that the impulse functions I_k, $k = 1, \dots, p$, are bounded. Then nonlinear impulsive problem* (NP) *has at least one solution.*

Corollary 2 *(Sublinear growth, Theorem 3.3 in [4]). Assume that there exist $a \in PC(J)$, $b \in R$, and $\alpha \in [0, 1)$ such that*

$$\left|f(t,u)\right| \leqslant a(t) + b|u|^{\alpha}, \quad t \in J,\ u \in R.$$

Moreover, suppose that there exist constants $a_k, b_k \in R$, $\alpha_k \in [0, 1)$, $k = 1, \dots, p$, with

$$\left|I_k(u)\right| \leqslant a_k + b_k|u|^{\alpha_k}, \quad u \in R.$$

Then the nonlinear problem (NP) *is solvable.*

Corollary 3. *Assume the following conditions hold.*

(i) $\lim_{|u|\to\infty} \frac{f(t,u)}{u} = m$, *where $m \neq \lambda$ is a constant,*
(ii) $\lim_{|u|\to\infty} \frac{I_k(u)}{u} = 0$.

Then the problem (NP) *has at least one solution.*

Proof. By condition (i), we have

$$\lim_{|u|\to\infty} \frac{f(t,u) - mu}{u} = 0. \tag{3.7}$$

Let $f(t,u) - mu = F(t,u)$ and $\lambda - m = \bar{\lambda}$. Hence Eq. (1.1) is reduced to

$$u'(t) + \bar{\lambda} u(t) = F\big(t, u(t)\big). \tag{3.8}$$

From (3.7) and condition (ii), using Theorem 3.4, we can conclude that the periodic boundary value problem (3.8)–(1.2)–(1.3) is solvable. Hence the problem (NP) has at least one solution. □

Example 2. Let $f(t,u) = e^{-u}\sin t$, $I_k(u) = u^{1/3}$. It follows that

$$\lim_{|u|\to\infty} \frac{f(t,u)}{u} = 0, \qquad \lim_{|u|\to\infty} \frac{I_k(u)}{u} = 0.$$

In light of Theorem 3.4, the nonlinear problem (NP) is solvable.

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